Area Spectrum of a Rotating Charged Black Hole Solution of Heterotic String Theory

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Abstract

The recent proposal of Maggiore that the periodicity of a black hole may be the origin of area quantization law is analized in the context of black holes in string theory. We use the period of motion of an outgoing wave, which is shown to be related to the vibrational frequency of the perturbed black hole, to quantize the horizon areas of a Sen black hole. It is shown that the equally spaced area spectrum takes the same form as the obtained by Zeng et. al. for Schwarzschild and Kerr black holes and the spacing is the same as that obtained through the quasinormal mode frequencies. In order to obtain this result, we do not need to use the small angular momentum assumption which is necessary in the quasinormal mode approach.

PACS: 04.70.Dy, 11.25-w, 97.60.Lf

Keywords: string theory, black holes, area spectrum

I. INTRODUCTION

Recently, investigation on quasinormal modes of black holes has shown the possibility to find black holes by detecting their astrophysical consequences [1] and also provided a way to check the string theory [2]. Additionaly, it was shown that, using Bohr's correspondence principle which states that transition frequencies at large quantum numbers are equal to classical oscillation frequencies, the quasinormal modes also provide a method to quantize the horizon area of black holes.

According to [3], the real part of quasinormal mode frequency is responsible for the area spectrum of black holes. Based on the quasinormal modes of Schwarzschild black hole in large n limit [4, 5], and later using the adiabatic invariant [6] they obtained the quantized horizon area

$$\Delta A = 32\pi M\delta M = 4\ln 3 l_p^2 \tag{1}$$

where M is the black hole mass and l_p is Planck's lenght (in units with $G = c = 1$). This idea stems from an analogy with the classical harmonic oscillator: the action integral of the form $A = \oint pdq$ for a quasiperiodic system is an adiabatic invariant in analytical mechanics. Specifically, the Hamiltonian of the one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega_c^2 q^2}{2} \tag{2}$$

and the corresponding adiabatic invariant is

$$A = \frac{E}{\omega_c}. \tag{3}$$

Therefore, in the black hole case the classical vibrational frequency ω_c and the system energy E are treated as the quasinormal mode frequencies ω and the mass M in the large n limit, giving

$$A = \int \frac{dM}{\omega}. \tag{4}$$

Maggiore [7] proposed that this treatment should be reexamined because the proper frequency of the equivalent harmonic oscillator contains real ω_R and imaginary ω_I contributions. Since the imaginary contribution is dominant for the highly excited quasinormal

modes, one should use the imaginary part of quasinormal mode frequencies to study the area spectrum of black holes. This modification gives in the case of Schwarzschild black hole, the quantized horizon area

$$\Delta A = 8\pi l_p^2 \tag{5}$$

and the origin of this area quantization appears to be the periodicity of the black hole in Euclidean time. The form of the horizon area quantized in units of l_p was proposed firstly by Bekenstein [8] and he also found that the horizon area of a non-extremal black hole is adiabatic invariant classically. Using the point particle model presented in [9], Bekenstein found that the smallest possible increase in horizon area of a non-extremal black hole is exactly $\Delta A = 8\pi l_p^2$.

Following the work of Maggiore, area spectrum of many black holes has been investigated as for example rotating metrics in [10, 11] and charged black holes in [12, 13]. There are also studies via quasinormal modes analysis in de Sitter spaces [14, 15], non-Einstein gravity [16–20] and other background spacetimes [21–25].

Banerjee et. al. [26], found that area spectrum of black holes can be obtained by computing the average squared energy of the outgoing wave in the view of quantum tunneling and recently, Zeng et. al. [27] employ the periodicity of outgoing wave to obtain area spectrum of Schwarzschild and Kerr black holes. Their idea is that for a perturbed black hole, the outgoing wave performs periodic motion outside the horizon and the corresponding period is related to the frequency of the outgoing wave. The gravity system in Kruskal coordinates is periodic with respect to Euclidean time and the motion of a particle in this periodic gravity system also owns a period given by the inverse of the Hawking temperature. Therefore they conclude that the frequency of the outgoing wave is given by the inverse of the Hawking temperature.

In this paper, we will follow the treatment of Zeng et. al. to obtain the area spectrum of the rotating charged black hole of heterotic string theory reported by Sen [28]. In order to obtain the area spectrum, we get the concrete value of vibration frequency by equaling the motion period of the outgoing wave to the period of the gravity system obtained when considering the Euclidean time.

II. THE SEN BLACK HOLE

Sen [28, 29] was able to find a charged, stationary, axially symmetric solution of the field equations by using target space duality, applied to the classical Kerr solution. The line element of this solution can be written, in generalized Boyer-Linquist coordinates, as

$$\begin{aligned} ds^2 = & -\left(1-\frac{2Mr}{\rho^2}\right)dt^2+\rho^2\left(\frac{dr^2}{\Delta}+d\theta^2\right)-\frac{4Mra\sin^2\theta}{\rho^2}dtd\varphi \\ & +\left(r\left(r+r_\alpha\right)+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2, \end{aligned} \tag{6}$$

where

$$\Delta = r\left(r+r_\alpha\right)-2Mr+a^2 \tag{7}$$

$$\rho^2 = r\left(r+r_\alpha\right)+a^2\cos^2\theta. \tag{8}$$

Here M is the mass of the black hole, $a=\frac{J}{M}$ is the specific angular momentum of the black hole and the electric charge is given by

$$r_\alpha = \frac{Q^2}{M}. \tag{9}$$

Note that in the particular case of a static black hole, i.e. $a=0$, the metric (6) coincides with the GMGHS solution [30] while in the particular case $r_\alpha=0$ it reconstructs the Kerr solution.

The Kerr-Sen space has a spherical event horizon, which is the biggest root of the equation $\Delta=0$ and is given by

$$r_H = \frac{2M-r_\alpha+\sqrt{\left(2M-r_\alpha\right)^2-4a^2}}{2}$$

or in terms of the black hole parameters M, Q and J,

$$r_H = M-\frac{Q^2}{2M}+\sqrt{\left(M-\frac{Q^2}{2M}\right)^2-\frac{J^2}{M^2}}. \tag{10}$$

The area of the event horizon is given by

$$A=\int\left(r_H^2+a^2\right)\sin\theta d\theta d\varphi=8\pi M\left(M-\frac{Q^2}{2M}+\sqrt{\left(M-\frac{Q^2}{2M}\right)^2-\frac{J^2}{M^2}}\right). \tag{11}$$

Equation (10) tell us that the horizon disappears unless

$$|J|\leq M^2-\frac{Q^2}{2},$$

therefore, the extremal black hole, $|J|=M^2-\frac{Q^2}{2}$, has $A=8\pi\,|J|$. The angular velocity at the horizon is given by

$$\Omega=\frac{J}{2M^2}\frac{1}{M-\frac{Q^2}{2M}+\sqrt{\left(M-\frac{Q^2}{2M}\right)^2-\frac{J^2}{M^2}}} \tag{12}$$

its electrostatic potential at the horizon can be written as

$$V=\frac{Q}{2M} \tag{13}$$

and the Hawking temperature is

$$T_H=\frac{\kappa_H\hbar}{2\pi}=\frac{\hbar\sqrt{\left(2M^2-Q^2\right)^2-4J^2}}{4\pi M\left(2M^2-Q^2+\sqrt{\left(2M^2-Q^2\right)^2-4J^2}\right)}. \tag{14}$$

For the Sen black hole, there is an ergosphere between the outer horizon and the infinite redshift surface. To avoid the dragging effect, one should perform the dragging coordinate transformation[31],

$$\phi=\varphi-\Omega t, \tag{15}$$

where the dragging angular velocity is given by (12). In this case, Eq. (6) takes the form

$$ds^2=-F\left(r\right)dt^2+\frac{1}{G\left(r\right)}dr^2+\rho^2d\theta^2+H^2\left(r\right)d\phi^2 \tag{16}$$

where

$$F\left(r\right)=\frac{\Delta\rho^2}{\left(r\left(r+r_\alpha\right)+a^2\right)^2-\Delta a^2\sin^2\theta} \tag{17}$$

$$G\left(r\right)=\frac{\Delta}{\rho^2} \tag{18}$$

$$H^2\left(r\right)=\frac{\sin^2\theta}{\rho^2}\left[\left(r\left(r+r_\alpha\right)+a^2\right)^2-\Delta a^2\sin^2\theta\right]. \tag{19}$$

III. AREA SPECTRUM OF A SEN BLACK HOLE

The Klein Gordon equation,

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi-\frac{m^2}{\hbar^2}\Phi=0, \tag{20}$$

gives the scalar field wave function Φ by using the metric 6. However, we can also obtain the wave function with the Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu S\partial_\nu S+m^2=0, \tag{21}$$

where the action S and the wave functionΦ are related by

$$\Phi=\exp[\frac{i}{\hbar}S(t,r,\theta,\phi)]. \tag{22}$$

Now, we will concentrate on using the Hamilton-Jacobi equation to find the wave function. In the dragging coordinate frame, the action S can be decomposed as [27, 31]

$$S(t,r,\theta)=-(E-m\Omega)t+W(r)+\Theta(\theta), \tag{23}$$

where E is the energy of the emitted particle measured by the observer at the infinity and m denotes the angular quantum number about ϕ. Using Eqs.(6) and (16), we find that near the horizon Θ vanishes and W can be solved as

$$W(r)=\frac{i\pi(E-m\Omega)}{\sqrt{F'(r_H)G'(r_H)}}, \tag{24}$$

where we only consider the outgoing wave and primes denote derivatives with respect to r. Hence, the wave function Φ can be written in the form

$$\Phi=\exp[-\frac{i}{\hbar}(E-m\Omega)t]\psi(r_H) \tag{25}$$

where $\psi(r_H)=\exp[-\frac{\pi(E-m\Omega)}{\sqrt{F'(r_H)G'(r_H)}}]$. Note that Φ is a periodic function with the period

$$T=\frac{2\pi}{\left(\frac{E-m\Omega}{\hbar}\right)} \tag{26}$$

or taking into account the relation $E = \hbar\omega$,

$$T = \frac{2\pi}{\left(\omega - \frac{m\Omega}{\hbar}\right)}. \tag{27}$$

It is well known that in Kruskal coordinates, the gravity system is a periodic system with respect to the Euclidean time. Hence, we will assume that particles moving in this background also own a period that has a geometric origin in the Hawking thermal radiation [32]. Therefore, the relation between T and T_H is

$$T = \frac{2\pi}{\kappa_H} = \frac{\hbar}{T_H}. \tag{28}$$

Based on Eq. (11), the change of horizon area of a Kerr black hole can be written as

$$\Delta A = 8\pi \left[\frac{\left(\sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2} + 2M^2 - Q^2\right)(2MdM - QdQ) - 2JdJ}{\sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2}} \right] \tag{29}$$

or using Eq. (14),

$$\Delta A = 8\pi \left[\frac{\hbar\,(2MdM - QdQ)}{4\pi M T_H} - \frac{2JdJ}{\sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2}} \right]. \tag{30}$$

Since M is the total mass of the black hole and using Eqs. (27) and (28), we can write

$$dM - VdQ = \hbar\omega = m\Omega + 2\pi T_H. \tag{31}$$

Substituting this relation into Eq.(30) gives

$$\Delta A = 8\pi \left[\hbar + \frac{\hbar m\Omega}{2\pi T_H} + \frac{\hbar V dQ}{2\pi T_H} - \frac{\hbar Q dQ}{4\pi M T_H} - \frac{2JdJ}{\sqrt{\left(2M^2 - Q^2\right)^2 - 4J^2}} \right] \tag{32}$$

and simplifying this expression using Eqs. (12) and (13), we finally obtain

$$\Delta A = 8\pi l_p^2.$$

Note that the equally spaced area spectrum of a Sen black hole obtained here is consistent with the result presented from the viewpoint of quasinormal modes for the GMGHS black hole in [12] and for the Kerr black hole in [10, 11, 27]. However, the small angular momentum limit, which is necessary from the perspective of quasinormal mode analysis, is not necessary to obtain the general area gap $8\pi l_p^2$.

IV. CONCLUSIONS

Although the quantum gravity theory has not been found, it is meaningful to investigate the quantum correction to the area spectrum. We use the new scheme proposed by Zeng et. al. [27] to quantize the horizon area of a charged rotating black hole of heterotic string theory. It was found that the period of the gravity system with respect to the Euclidean time can determine the area spectrum of black holes. This result confirms the speculation of Maggiore that the periodicity of a black hole may be the origin of the area quantization. It is important to note that this approach is more convenient and simple to apply to the Sen metric since the quasinormal mode frequency could lead to some confusion on whether the real part or imaginary part is responsible for the area spectrum.

Acknowledgements

This work was supported by the Universidad Nacional de Colombia. Hermes Project Code 13038.

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